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Colliding Wave Solutions, Duality, and Diagonal Embedding of General Relativity in Two-Dimensional Heterotic String Theory

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Abstract

The non-linear sigma model of the dimensionally reduced Einstein (-Maxwell) theory is diagonally embedded into that of the two-dimensional heterotic string theory. Consequently, the embedded string backgrounds satisfy the (electro-magnetic) Ernst equation. In the pure Einstein theory, the Matzner-Misner SL(2,**R**) transformation can be viewed as a change of conformal structure of the compactified flat two-torus, and in particular its integral subgroup SL(2,**Z**) acts as the modular transformation. The Ehlers SL(2,**R**) and SL(2,**Z**) similarly act on another torus whose conformal structure is induced through the Kramer-Neugebauer involution. Either of the Matzner-Misner and the Ehlers SL(2,**Z**) can be embedded to a special T-duality, and if the former is chosen, then the Ehlers SL(2,**Z**) is shown to act as the S-duality on the four-dimensional sector. As an application we obtain some new colliding string wave solutions by using this embedding as well as the inverse scattering method.

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I. INTRODUCTION

Recently duality symmetries have attracted much attention in string theory and its low-energy effective field theory. The symmetries of interest consist of two different types of discrete groups — T- and S- duality groups. The former reflects an automorphism of the target space (See [1] for a review.), while the latter is a conjectured non-perturbative weak-strong coupling duality symmetry [2] in the sense of the string-loop expansion. It has been argued that this may also be an exact symmetry of a full string theory when the theory has enough ($N = 4$) supersymmetries so that a certain kind of non-renormalization theorem may ensure to extend the tree-level result to the full theory (See [3,4]).

A characteristic feature of the duality symmetries is that they are discrete subgroups of non-compact symmetries of effective supergravity theories known for a long time. For example, in the toroidal compactification to four dimensions the low-energy effective action of heterotic string theory [5] not only exhibits a manifest O(6,22), but also possesses an SL(2,$\mathbf{R}$) symmetry [6]. The Dynkin diagram of this group gets an additional node to be O(8,24) in the further reduction to three dimensions [7,8], and, in accord with the general rule [9] of the dimensional reduction of nonlinear sigma models, gets one more in two dimensions to be the loop group $\widehat{\mathrm{O}}$(8,24), in which the duality group is embedded [10]. The integrable supergravity theories arising in this way have already been studied in detail in Refs. [11,12]. The type II string version, which fits in the "hidden" E_7 symmetry [13] in the case of four dimensions, was also investigated and called "U-duality" [14].

The similarity between the appearance of hidden symmetries in supergravity theories and the Geroch group [15] in general relativity was first pointed out in Ref. [9], and was elaborated in [16]. It is well-known that the reduced Einstein gravity to two dimensions possesses (in the infinitesimal form) the affine $\widehat{\mathrm{sl}}$(2,$\mathbf{R}$) Kac-Moody symmetry with a non-trivial central term [9]. It is generated by two fundamental finite sl(2,$\mathbf{R}$) subalgebras, which are called the Matzner-Misner [17] and the Ehlers [18] sl(2,$\mathbf{R}$), respectively. The Matzner-Misner SL(2,$\mathbf{R}$) — a general-relativity analogue of the T-duality (if restricted to its integral

discrete subgroup)— is a symmetry of the SL(2,**R**)/U(1) nonlinear sigma model obtained in the dimensional reduction of the Einstein action directly from four to two dimensions. The Ehlers SL(2,**R**) — an analogue of the S-duality — is a symmetry of another SL(2,**R**)/U(1) nonlinear sigma model which is constructed by trading the Kaluza-Klein vector field in three dimensions with the axion-like "twist potential". The purpose of this paper is to make the relation between the two theories more concrete by "diagonally" embedding the two-dimensional nonlinear sigma model associated with the dimensional reduction of the Einstein(-Maxwell) theory into that of the two-dimensional effective heterotic string theory compactified on an eight-torus.

It is obvious that such an embedding is possible. Specifically, we will diagonally embed the SL(2,**R**) as a subgroup of $\mathrm{O}(8,8) \supset \mathrm{O}(2,2) \otimes I_4$ (where I_n denotes the n-dimensional identity matrix), and further the SU(2,1), the symmetry group of the reduced Einstein-Maxwell theory, as a subgroup of $\mathrm{O}(8,24) \supset \mathrm{O}(2,6) \otimes I_4$. We consider this embedding as we are motivated by the following two practical relevances. First, by this embedding we are endowed with a simple picture of the T- and S- duality symmetries on the particular string backgrounds. We will show that the integral Matzner-Misner SL(2,**Z**) transformation becomes a special T-duality, while the integral Ehlers SL(2,**Z**) acts as the S-duality on the four-dimensional sector, if they are embedded into the string theory. On the other hand, it can be shown that the Matzner-Misner SL(2,**R**) transformation is a change of conformal structure of the "compactified" flat two-torus, and in particular the integral ones are the modular transformations. Correspondingly, the Ehlers SL(2,**R**) similarly changes the conformal structure of another torus induced through the Kramer-Neugebauer involution. Thus it provides us a simple picture of the T- and S- duality groups acting on the embedded configurations. Second, more importantly, since the embedding is diagonal, the sigma model Lagrangian of the effective two-dimensional heterotic string on the embedded special backgrounds takes completely the same form as that of the reduced Einstein(-Maxwell) theory. This means that the embedded string backgrounds satisfy the (electro-magnetic) Ernst equation. Hence one may recast every known solution of the (electro-magnetic) Ernst

equation into that of the corresponding string background. We would like to emphasize here the difference between our method and the solution-generating techniques applied to string theory in the previous litrature, in the latter of which one takes a known solution of the Einstein theory as a *trivial* solution of string theory with all other fields set to zero, and then apply the duality transformations to it to obtain a non-trivial solution. In contrast, the degrees of freedoms of the Kaluza-Klein vector and the Maxwell fields are related to, respectively, the anti-symmetric tensor and the abelian gauge fields in our diagonal embedding, which enable us to read off a string solution directly from a solution of general relativity. At the same time the new solutions themselves obtained in this way can be "seeds" solutions to which the duality transformations apply. In view of the vast accumulation of knowledge of the solution-generating technique in general relativity, we hope that this analogy may provide deeper insight on the duality symmetry of string theory. Although we only consider the colliding wave solitions in this paper, our discussion can be straightforwardly applied to the case of the (++) signature.

The plan of this paper is as follows. In Sect.2 we will briefly review the basic facts on the dimensionally reduced Einstein gravity in the presence of two (space-like) commuting Killing vector fields, and then explain the geometrical picture of the Matzner-Misner and the Ehlers symmetry. In Sect.3 we apply the inverse scattering method to the two-dimensional effective heterotic string theory, developing the formulation of the linear system obtained in Ref. [10]. The construction of a sample solution motivates us to consider the diagonal embedding in the subsequent sections. In Sect.4 we will embed the nonlinear sigma models of the reduced Einstein and the Einstein-Maxwell theories into the effective string theory without and with the $\mathrm{U}(1)^{16}$ gauge fields, respectively. We will see that either of the integral Matzner-Misner and Ehlers SL(2,**Z**) can be embedded into a special T-duality because of the Kramer-Neugebauer involution, and that, in particular, if the Matzner-Misner one is identified as a T-duality, then the Ehlers one acts as the S-duality on the four dimensional sector. In Sect.5 we will obtain some new colliding string wave solutions by simply recasting the Ferrari-Ibañez infinite series of metrics, the Nutku-Halil metric, and its electro-magnetic

generalization. Finally we summarize our results in Sect.6.

II. DIMENSIONALLY REDUCED EINSTEIN GRAVITY

A. Ehlers and Matzner-Misner SL(2,R)

As have been already stated in Introduction, the dimensionally reduced Einstein gravity is the simplest prototype that exhibits a non-compact "hidden" symmetry. We first consider the reduction of the Einstein action from four dimensions to three by introducing a Killing vector field along the $x^3 \equiv y$ axis (In this section we drop the $x^2 \equiv x$ and $x^3 \equiv y$ dependences while leave x^0 and x^1 as the coordinates of the reduced two-dimensional field theory.).

We start from the following vierbein

$$E_M^{\ A} = \begin{bmatrix} \Delta^{-\frac{1}{2}} e_{\bar{\mu}}^{\ \bar{\alpha}} & \Delta^{\frac{1}{2}} B_{\bar{\mu}} \\ 0 & \Delta^{\frac{1}{2}} \end{bmatrix}, \tag{1}$$

where M and A denote the four-dimensional spacetime and Lorentz indices, while $\bar{\mu}$ and $\bar{\alpha}$ do the corresponding three-dimensional ones. All the components are assumed to be independent of the y coordinate. Up to a total derivative, the Einstein Lagrangian is reduced to

$$\mathcal{L} = \sqrt{-\mathrm{det} E_M^{\ A}}\, R(E_M^{\ A}) \tag{2}$$

$$= \sqrt{-g^{(3)}} \left[R^{(3)} - \frac{1}{4}\Delta^2 F_{\bar{\mu}\bar{\nu}} F^{\bar{\mu}\bar{\nu}} - \frac{1}{2} g^{(3)\bar{\mu}\bar{\nu}} \Delta^{-2} \partial_{\bar{\mu}}\Delta \partial_{\bar{\nu}}\Delta \right], \tag{3}$$

where $g^{(3)}_{\bar{\mu}\bar{\nu}} = e_{\bar{\mu}}^{\ \bar{\alpha}} \eta_{\bar{\alpha}\bar{\beta}} e^{\bar{\beta}}_{\ \ \bar{\nu}}$, $F_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\mu}} B_{\bar{\nu}} - \partial_{\bar{\nu}} B_{\bar{\mu}}$ and $F^{\bar{\mu}\bar{\nu}} = g^{(3)\bar{\mu}\bar{\sigma}} g^{(3)\bar{\nu}\bar{\lambda}} F_{\bar{\sigma}\bar{\lambda}}$.

The Ehlers symmetry is not a symmetry of the original Einstein action, but a symmetry of the equations of motion, as other S-like duality symmetries are. It is, however, known in supergravity theories that there exist a trick to obtain an action manifestly invariant under such symmetries. Namely we add a Lagrange multiplier term to the Lagrangian to treat $F^{\bar{\mu}\bar{\nu}}$ as an independent field[1]

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[1] Here $\epsilon^{\bar{\mu}\bar{\nu}\bar{\sigma}}$ is the "densitized" totally anti-symmetric tensor, while the "undensitized" one, which

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2} B \epsilon^{\bar{\mu}\bar{\nu}\bar{\sigma}} \partial_{\bar{\sigma}} F_{\bar{\mu}\bar{\nu}}. \tag{4}$$

The equation of motion of $F^{\bar{\mu}\bar{\nu}}$ reads

$$\Delta^2 F^{\bar{\mu}\bar{\nu}} = (\sqrt{-g^{(3)}})^{-1} \epsilon^{\bar{\mu}\bar{\nu}\bar{\sigma}} \partial_{\bar{\sigma}} B. \tag{5}$$

Substituting (5) into (4), we obtain an SL(2,R)/U(1) non-linear sigma model coupled to gravity

$$\mathcal{L}' = \sqrt{-g^{(3)}} \left[R^{(3)} - \frac{1}{2} g^{(3)\bar{\mu}\bar{\nu}} \Delta^{-2} (\partial_{\bar{\mu}} B \partial_{\bar{\nu}} B + \partial_{\bar{\mu}} \Delta \partial_{\bar{\nu}} \Delta) \right], \tag{6}$$

which is manifestly invariant under the fractional linear transformation

$$Z^{(\mathrm{E})} \to \frac{a Z^{(\mathrm{E})} + b}{c Z^{(\mathrm{E})} + d} \quad (a, b, c, d \in \mathbf{R}), \tag{7}$$

where

$$Z^{(\mathrm{E})} = B + i\Delta. \tag{8}$$

This is the Ehlers SL(2,$\mathbf{R}$) symmetry (One may set $ad - bc = 1$ without loss of generality.) [18]. $Z^{(\mathrm{E})}$ is related to the Ernst potential $\mathcal{E}$ by

$$\mathcal{E} = i\overline{Z^{(\mathrm{E})}}. \tag{9}$$

We descend further from three to two dimensions by discarding the $x^2 \equiv x$-coordinate dependence of the fields. Assuming the form of the vierbein as

$$E_M{}^A = \left[\begin{array}{ccc} \Delta^{-\frac{1}{2}} \lambda \delta_\mu{}^\alpha & \multicolumn{2}{c}{0} \\ 0 & \Delta^{-\frac{1}{2}} \rho & \Delta^{\frac{1}{2}} B_x \\ & 0 & \Delta^{\frac{1}{2}} \end{array} \right], \tag{10}$$

where μ and ν stand for the two-dimensional spacetime and Lorentz indices, the reduced two dimensional Lagrangian is found to be

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takes the values $\pm(\sqrt{-g^{(3)}})^{-1}$, has been used in some literature.

$$\mathcal{L}' = \rho\eta^{\mu\nu}\left[-2\partial_\mu\partial_\nu \ln\lambda - \frac{1}{2}\Delta^{-2}(\partial_\mu\Delta\partial_\nu\Delta + \partial_\mu B\partial_\nu B)\right]. \tag{11}$$

The indices $\mu, \nu = 0, 1$ are now raised by $\eta^{\mu\nu}$. The relation (5) then becomes

$$\Delta^2\partial_\mu B_x = \rho\epsilon^{(2)}{}_\mu{}^\nu\partial_\nu B, \quad \epsilon^{(2)}{}_\mu{}^\nu = \pm 1 \tag{12}$$

in two dimensions. The twist potential B and a component of the vierbein B_x are related non-locally with each other by the equation (12). The equations of motions of Δ and B are compactly described by the Ernst equation [19]

$$\rho\partial_\mu\mathcal{E}\partial^\mu\mathcal{E} = \Delta\partial_\mu(\rho\partial^\mu\mathcal{E}). \tag{13}$$

Another way to obtain a two-dimensional model is to reduce the dimensions from four to two directly. Substituting the parameterization (10) in (2), we find

$$\mathcal{L} = \rho\eta^{\mu\nu}\left[-2\partial_\mu\partial_\nu\ln(\lambda\Delta^{-\frac{1}{2}}\rho^{\frac{1}{4}}) - \frac{\Delta^2}{2\rho^2}\left\{\partial_\mu\left(\frac{\rho}{\Delta}\right)\partial_\nu\left(\frac{\rho}{\Delta}\right) + \partial_\mu B_x\partial_\nu B_x\right\}\right]. \tag{14}$$

Defining

$$Z^{(\mathrm{MM})} = B_x + i\frac{\rho}{\Delta} \tag{15}$$

in this case, the Lagrangian (14) is invariant under the Matzner-Misner SL(2,R) transformation

$$Z^{(\mathrm{MM})} \to \frac{aZ^{(\mathrm{MM})} + b}{cZ^{(\mathrm{MM})} + d} \quad (a, b, c, d \in \mathbf{R};\ ad - bc = 1). \tag{16}$$

The two SL(2,**R**) groups (7) and (16) generate the Geroch group. It is well-known that they are nontrivially entangled with each other to constitute an infinite-dimensional sl(2,**R**) Kac-Moody algebra [9]. The replacement of the variables

$$B \leftrightarrow B_x, \quad \Delta \leftrightarrow \frac{\rho}{\Delta}, \quad \lambda \leftrightarrow \lambda\Delta^{-\frac{1}{2}}\rho^{\frac{1}{4}}, \quad \rho \leftrightarrow \rho, \tag{17}$$

which makes the Lagrangians (11) and (14) identical, is called the Kramer-Neugebauer involution [20]. Let $\mathcal{E}^{(\mathrm{MM})}$ be the image of $\mathcal{E}$ by (17), then obviously $\mathcal{E}^{(\mathrm{MM})}$ also satisfies the "Ernst equation"

$$\rho\partial_\mu\mathcal{E}^{(\mathrm{MM})}\partial^\mu\mathcal{E}^{(\mathrm{MM})} = \frac{\rho}{\Delta}\partial_\mu(\rho\partial^\mu\mathcal{E}^{(\mathrm{MM})}). \tag{18}$$

B. Integral Matzner-Misner SL(2,Z) as the Modular Transformation of a Torus

It is not an accident that the Teichmüller space of a torus (= the upper-half plane) appears as the target space of the sigma model of the dimensionally reduced Lagrangian (14). To clarify this point, let us assume that the two commuting Killing vector fields are tangent to a two-torus. In other words we consider the "toroidal compactification" of Einstein gravity. The resulting action of the reduced theory depends only on the two coordinates x^μ, $\mu = 0, 1$, and each point x^μ associates a flat torus parameterized by the coordinates x^m, $m = 2, 3$. The constant metric of this torus is given by the zweibein

$$E_m^{\ a} = \begin{bmatrix} \Delta^{-\frac{1}{2}}\rho & \Delta^{\frac{1}{2}}B_x \\ 0 & \Delta^{\frac{1}{2}} \end{bmatrix}. \tag{19}$$

The line element reads

$$dL^2 = \Delta\left[(dy + B_x dx)^2 + (\frac{\rho}{\Delta}dx)^2\right] \quad (x^2 \equiv x, x^3 \equiv y). \tag{20}$$

It is well-known that each point $z = \tau$ in the upper-half complex plane represents an inequivalent complex structure of a torus. For the above metric (20) it turns out that the modular parameter τ coincides with $Z^{(\mathrm{MM})}$. Indeed, this can be checked as follows: τ ($\mathrm{Im}\tau > 0$) stands for a constant metric of a torus obtained as a lattice on the complex plane, whose periods are 1 and τ. Mapping this rectangle to a square $0 \le x \le 1$, $0 \le y \le 1$, the metric becomes

$$dL^2 = (dy + \mathrm{Re}\tau dx)^2 + (\mathrm{Im}\tau dx)^2. \tag{21}$$

Comparing (20) with (21), we find that the conformal structure of the metric (20) is represented by the modular parameter

$$\tau = B_x + i\frac{\rho}{\Delta} \tag{22}$$

(Note that all Δ, ρ and B_x are constant with respect to the coodinates of the torus x, y.).

It is now easy to see the meaning of the Matzner-Misner symmetry; this is nothing but the invariance of the system under the change of conformal structure of the compactified

torus. Of course, an infinitesimal change can be a symmetry because we consider a Kaluza-Klein theory, i.e. only the constant modes on the compactified space. Therefore, if we consider the full theory in which all the massive excitations are included, then we may only have the discrete modular group symmetry

$$Z^{(\mathrm{MM})} \rightarrow \frac{aZ^{(\mathrm{MM})}+b}{cZ^{(\mathrm{MM})}+d} \quad (a,b,c,d \in \mathbf{Z};\ ad-bc=1). \tag{23}$$

We will see in the next subsection that this is a special T duality if embedded in string theory.

What is the meaning of the Ehlers symmetry, then? We can see this by using the Kramer-Neugebauer involution, by which the zweibein (19) is mapped to

$$E_m{}^a = \left(\frac{\rho}{\Delta}\right)^{\frac{1}{2}} \begin{bmatrix} \Delta & B \\ 0 & 1 \end{bmatrix}. \tag{24}$$

The line element reads

$$dL^2 = \frac{\rho}{\Delta}\left[(dy+Bdx)^2 + (\Delta dx)^2\right]. \tag{25}$$

Again it is easy to see that $Z^{(\mathrm{E})}$ coincides to the modular parameter of the metric (24), but in this case this metric is the one on a "fictitious" torus; one of the component B is related non-locally to a component B_x of a "real" torus. The Ernst potential $\mathcal{E}$ is then essentially a modular parameter of this torus[2]. The Ehlers integral fractional linear transformation

$$Z^{(\mathrm{E})} \rightarrow \frac{aZ^{(\mathrm{E})}+b}{cZ^{(\mathrm{E})}+d} \quad (a,b,c,d \in \mathbf{Z}), \tag{26}$$

is then the modular transformation of this torus. In particular, the modular S transformation[3]

[2] In the Ashtekar formulation of the dimensionally reduced gravity, the Ashtekar connections corresponding to the compactified direction were shown to be modular forms [21].

[3] Of course this confusing name has nothing to do with the fact, which we will show later, that the Ehlers SL(2,$\mathbf{Z}$) acts as the S duality.

$$Z^{(\mathrm{E})} \to -\frac{1}{Z^{(\mathrm{E})}}, \tag{27}$$

which is a special element of the Ehlers SL(2,**Z**) group, is simply referred to as the "Ehlers transformation" in the literature.

III. LINEAR SYSTEM AND INVERSE SCATTERING METHOD

A. Linear System

In this section we first review the construction of the linear system of the two-dimensional heterotic string theory [10], and then use this formulation to obtain a classical solution by means of the inverse scattering method. We will closely follow the Ref. [11].

It has been shown [22] that the bosonic sector of the ten-dimensional effective low-energy action of the heterotic string theory

$$\begin{aligned}
S = \int d^{10}z\sqrt{-G^{(10)}}e^{-\Phi^{(10)}}\Big(&R^{(10)} + G^{(10)MN}\partial_M\Phi^{(10)}\partial_N\Phi^{(10)} \\
&- \frac{1}{12}H^{(10)}_{MNP}H^{(10)MNP} - \frac{1}{4}F^{(10)I}_{MN}F^{(10)IMN}\Big),
\end{aligned} \tag{28}$$

$$F^{(10)I}_{MN} = \partial_M A^{(10)I}_N - \partial_N A^{(10)I}_M, \tag{29}$$

$$H^{(10)}_{MNP} = \left(\partial_M B^{(10)}_{NP} - \frac{1}{2}A^{(10)I}_M F^{(10)I}_{NP}\right) + (\text{cyclic permutations}),$$
$$(M, N, P = 0, \ldots, 9)$$

is reduced to the two dimensional action

$$S = \int d^2x\sqrt{-G}e^{-\Phi}\left[R_G + G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{8}G^{\mu\nu}\mathrm{Tr}(\partial_\mu ML\partial_\nu ML)\right] \tag{30}$$

if compactified on an eight-torus, where the ten-dimensional metric $G^{(10)}_{MN}$, anti-symmetric tensor field $B^{(10)}_{MN}$, $\mathrm{U}(1)^{16}$ gauge field $A^{(10)I}_M$, and the dilaton field $\Phi^{(10)}$ are assumed to take the forms

$$G^{(10)}_{MN} = \begin{bmatrix} G_{\mu\nu} & 0 \\ 0 & \widehat{G}_{mn} \end{bmatrix},\ B^{(10)}_{MN} = \begin{bmatrix} B_{\mu\nu} = 0 & 0 \\ 0 & \widehat{B}_{mn} \end{bmatrix},$$

$$A_M^{(10)I} = \begin{bmatrix} A_\mu^I = 0 \\ \widehat{A}_m^I \end{bmatrix}, \ e^{-\Phi^{(10)}} = (\det \widehat{G})^{-\frac{1}{2}} e^{-\Phi}. \tag{31}$$

In (31) some fields without physical degrees of freedom in two dimensions have been set to zero. x^μ ($\mu = 0, 1$) and y^m ($m = 1, \ldots, 8$) are the coordinates of the two dimensional space-time and the eight-dimensional compactified torus, respectively. The 32×32 matrices M and L are given by

$$M = \begin{bmatrix} \widehat{G}^{-1} & \widehat{G}^{-1}(\widehat{B} + \widehat{C}) & \widehat{G}^{-1}\widehat{A} \\ (-\widehat{B} + \widehat{C})\widehat{G}^{-1} & (\widehat{G} - \widehat{B} + \widehat{C})\widehat{G}^{-1}(\widehat{G} + \widehat{B} + \widehat{C}) & (\widehat{G} - \widehat{B} + \widehat{C})\widehat{G}^{-1}\widehat{A} \\ \widehat{A}^T\widehat{G}^{-1} & \widehat{A}^T\widehat{G}^{-1}(\widehat{G} + \widehat{B} + \widehat{C}) & I_{16} + \widehat{A}^T\widehat{G}^{-1}\widehat{A} \end{bmatrix} \tag{32}$$

with $\widehat{C} = \frac{1}{2}\widehat{A}\widehat{A}^T$, and

$$L = \begin{bmatrix} & I_8 & \\ I_8 & & \\ & & -I_{16} \end{bmatrix}. \tag{33}$$

They satisfy $M^T = M$, $MLM^T = L$. We further define

$$U_0 = \begin{bmatrix} \frac{1}{\sqrt{2}}I_8 & -\frac{1}{\sqrt{2}}I_8 & \\ \frac{1}{\sqrt{2}}I_8 & \frac{1}{\sqrt{2}}I_8 & \\ & & I_{16} \end{bmatrix} \tag{34}$$

and

$$U_0^T M U_0 \equiv M'. \tag{35}$$

M' is again a symmetric matrix $M'^T = M'$, and

$$U_0^T L U_0 = \begin{bmatrix} I_8 & & \\ & -I_8 & \\ & & -I_{16} \end{bmatrix} \equiv I_{8,24}. \tag{36}$$

M' belongs to O(8,24), preserving the bilinear form defined by $I_{8,24}$ invariant

$$M'^T I_{8,24} M' = I_{8,24}. \tag{37}$$

The Lie algebra o(8,24) consists of the matrices

$$\left\{ \left[\begin{array}{cc} X_1 & X_2 \\ X_2^T & X_3 \end{array} \right] \middle| \; X_1^T = -X_1, \quad X_3^T = -X_3 \right\}. \tag{38}$$

X_2 is an arbitrary real 8×24 matrix. The symmetric-space automorphism τ of the coset O(8,24)/O(8)×O(24) is given by [23]

$$\tau(X) = I_{8,24} X I_{8,24} \quad (X \in \mathrm{o}(8{,}24)). \tag{39}$$

o(8,24) is decomposed into eigenspaces of τ as

$$\mathrm{o}(8{,}24) = \mathbf{K} \oplus \mathbf{H}, \tag{40}$$

$$\mathbf{K} = \left\{ \left[\begin{array}{cc} 0 & X_2 \\ X_2^T & 0 \end{array} \right] \middle| \; X_2 : \text{arbitrary real } 8 \times 24 \text{ matrix} \right\}, \tag{41}$$

$$\mathbf{H} = \left\{ \left[\begin{array}{cc} X_1 & 0 \\ 0 & X_3 \end{array} \right] \middle| \; X_1^T = -X_1, \; X_3^T = -X_3 \right\}, \tag{42}$$

so that $\tau(\mathbf{K}) = -\mathbf{K}$, $\tau(\mathbf{H}) = \mathbf{H}$. $\mathbf{H}$ is the Lie algebra of the denominator O(8)×O(24) of the coset. Since $X^T = X$ if $X \in \mathbf{K}$, and $X^T = -X$ if $X \in \mathbf{H}$, (39) is equivalent to

$$\tau(g) = (g^T)^{-1} \quad (g \in \mathrm{O}(8{,}24)). \tag{43}$$

It is known that M' can be written as

$$M' = \mathcal{V}\mathcal{V}^T, \tag{44}$$

where $\mathcal{V} \in$ O(8,24) is given by [22]

$$\mathcal{V} = U_0^T V U_0, \quad V = \left[\begin{array}{ccc} \widehat{E}^{-1} & 0 & 0 \\ (-\widehat{B} + \widehat{C})\widehat{E}^{-1} & \widehat{E}^T & \widehat{A} \\ \widehat{A}^T \widehat{E}^{-1} & 0 & I_{16} \end{array} \right]. \tag{45}$$

$\widehat{E}$ is an "achtbein" (vielbein with eight indices) of $\widehat{G}$, satisfying $\widehat{G} = \widehat{E}^T \widehat{E}$. Using the decomposition (40) we write

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} = P_\mu + Q_\mu, \quad P_\mu \in \mathbf{K}, \quad Q_\mu \in \mathbf{H}, \tag{46}$$

then

$$\begin{aligned}
&\mathrm{Tr}(\partial_\mu M L \partial_\nu M L) \\
&= -\mathrm{Tr}(M^{-1}\partial_\mu M \cdot M^{-1}\partial_\nu M) \\
&= -\mathrm{Tr}(M'^{-1}\partial_\mu M' \cdot M'^{-1}\partial_\nu M') \\
&= -\mathrm{Tr}\left[\left(\mathcal{V}^{-1}\partial_\mu \mathcal{V} + (\mathcal{V}^{-1}\partial_\mu \mathcal{V})^T\right) \cdot \left(\mathcal{V}^{-1}\partial_\nu \mathcal{V} + (\mathcal{V}^{-1}\partial_\nu \mathcal{V})^T\right)\right] \\
&= -4\mathrm{Tr} P_\mu P_\nu.
\end{aligned} \tag{47}$$

In the last line we used $P_\mu^T = P_\mu$ and $Q_\mu^T = -Q_\mu$. The O(8)×O(24) gauge transformation $\mathcal{V} \mapsto \mathcal{V}h(x)$, $h(x) \in$ O(8)×O(24), which does not change M', acts on P_μ and Q_μ as

$$P_\mu \mapsto h^{-1}P_\mu h, \quad Q_\mu \mapsto h^{-1}Q_\mu h + h^{-1}\partial_\mu h. \tag{48}$$

Hence (47) is manifestly gauge invariant. The dilaton kinetic term can be absorbed into the R_G term by a rescaling

$$G_{\mu\nu} = e^{\Phi} g_{\mu\nu}, \tag{49}$$

so that the action becomes

$$S = \int d^2x e^{-\Phi}\left[\sqrt{-g}R_g - \frac{1}{2}\eta^{\mu\nu}\mathrm{Tr} P_\mu P_\nu\right], \tag{50}$$

where we have adopted the conformal gauge

$$g_{\mu\nu} = \lambda^2 \eta_{\mu\nu}. \tag{51}$$

This is precisely the same Lagrangian resulting from the reduction of the Einstein Lagrangian from four dimensions to two, except the difference of the compactified sector. This observation is a clue to construct a mapping from solutions of general relativity to those of string theory. The independent equations of motion arising from the action (50) are

$$\partial_+\partial_- e^{-\Phi} = 0, \tag{52}$$

$$\lambda^{-1}\partial_\pm\lambda\cdot\rho\partial_\pm\rho = \frac{1}{2}\rho^{-1}\partial_\pm^2\rho + \frac{1}{4}\mathrm{Tr}P_\pm P_\pm, \tag{53}$$

$$D_+(\rho P_-) + D_-(\rho P_+) = 0, \tag{54}$$

where we switched to the light-cone coordinate, and

$$D_\mu P_\nu \equiv \partial_\mu P_\nu + [Q_\mu,\ P_\nu]. \tag{55}$$

We next introduce the spectral-parameter dependent matrix $\widehat{\mathcal{V}}$ [24,16] defined by

$$\widehat{\mathcal{V}}^{-1}\partial_\mu\widehat{\mathcal{V}} = Q_\mu + \frac{1+t^2}{1-t^2}P_\mu + \frac{2t}{1-t^2}\epsilon_{\mu\nu}P^\nu. \tag{56}$$

The zero-curvature condition

$$\partial_\mu(\widehat{\mathcal{V}}^{-1}\partial_\nu\widehat{\mathcal{V}}) - \partial_\nu(\widehat{\mathcal{V}}^{-1}\partial_\mu\widehat{\mathcal{V}}) + \left[\widehat{\mathcal{V}}^{-1}\partial_\mu\widehat{\mathcal{V}},\ \widehat{\mathcal{V}}^{-1}\partial_\nu\widehat{\mathcal{V}}\right] = 0 \tag{57}$$

is equivalent to the relevant equation of motion (54) and the "flat-space" zero-curvature condition (for $\mathcal{V} = \widehat{\mathcal{V}}(t=0)$) [25], if the spectral parameter t satisfies [24,16,11]

$$\partial_\pm \ln\rho = \frac{1\mp t}{1\pm t}\partial_\pm \ln t. \tag{58}$$

This equation can be integrated to give

$$w = \frac{1}{4}(\rho_+(x^+) + \rho(x^-))\left(t + \frac{1}{t}\right) - \frac{1}{2}(\rho_+(x^+) - \rho(x^-)) \tag{59}$$

with some real constant w, where we solved the equation of motion of the dilaton Φ to decompose it into left- and right-moving components

$$e^{-\Phi} \equiv \rho \equiv \rho_+(x^+) + \rho(x^-). \tag{60}$$

B. Inverse Scattering Method

To demonstrate how this formulation works, let us construct the simplest colliding string wave solution. In our O(8,24)/O(8)×O(24) case, the generalized symmetric-space automorphism τ^∞ turns out to be

$$
\begin{aligned}
\tau^{\infty}\left(\widehat{\mathcal{V}}(t)\right) &\equiv \tau(\widehat{\mathcal{V}}(t^{-1})) \\
&= I_{8,24}\widehat{\mathcal{V}}(t^{-1})I_{8,24} \\
&= \left[(\widehat{\mathcal{V}}(t^{-1}))^T\right]^{-1}.
\end{aligned} \tag{61}
$$

This has the following properties

$$
P_\mu \to -P_\mu, \quad Q_\mu \to Q_\mu, \quad t \to t^{-1}, \tag{62}
$$

and

$$
\tau^{\infty}\left(\widehat{\mathcal{V}}^{-1}\partial_\mu\widehat{\mathcal{V}}\right) = \widehat{\mathcal{V}}^{-1}\partial_\mu\widehat{\mathcal{V}}. \tag{63}
$$

Owing to the invariance (63), the monodromy matrix

$$
\mathcal{M} \equiv \widehat{\mathcal{V}}\tau^{\infty}(\widehat{\mathcal{V}}^{-1}) \tag{64}
$$

does not depend on x^μ. Hence one may construct a solution by finding $\mathcal{M}$ that factorizes into $\widehat{\mathcal{V}}$ and $\tau^{\infty}(\widehat{\mathcal{V}}^{-1})$, and then calculating the conformal factor λ^2 for such $\widehat{\mathcal{V}}$.

Motivated by the example discussed in Ref. [11], let us take the following monodromy matrix

$$
\mathcal{M} = U_0^T \begin{bmatrix} -\frac{w-\frac{1}{2}}{w+\frac{1}{2}}I_8 & & \\ & -\frac{w+\frac{1}{2}}{w-\frac{1}{2}}I_8 & \\ & & 0 \end{bmatrix} U_0. \tag{65}
$$

Fixing the conformal invariance by choosing

$$
\rho_+ = \frac{1}{2} - (x^+)^2 \equiv \frac{1}{2} - u^2, \quad \rho_- = \frac{1}{2} - (x^-)^2 \equiv \frac{1}{2} - v^2, \tag{66}
$$

it is straightforward to check that this choice of $\mathcal{M}$ allows the factorization (64) as follows:

$$
\widehat{\mathcal{V}} = U_0^T \begin{bmatrix} -\sqrt{-\frac{t_2}{t_1}\frac{t-t_1}{t-t_2}}I_8 & & \\ & -\sqrt{-\frac{t_1}{t_2}\frac{t-t_2}{t-t_1}}I_8 & \\ & & 0 \end{bmatrix} U_0, \tag{67}
$$

$$t_1 = \frac{\sqrt{1-u^2}-v}{\sqrt{1-u^2}+v}, \quad t_2 = \frac{u+\sqrt{1-v^2}}{u-\sqrt{1-v^2}}. \tag{68}$$

($-\frac{t_2}{t_1} \in \mathbf{R}$, > 0 in the interaction region $u, v > 0$ and $u^2+v^2<1$.) Note that the $\mathcal{M}$ matrix (65) is factorized in the same way as is done in the general relativity case, because of the formal reminiscence of τ^∞ (61). From (45) we find that this choice of $\mathcal{M}$ corresponds to the configurations

$$\widehat{G}_{mn} = -\frac{t_2}{t_1}\delta_{mn}, \quad \widehat{B}_{mn} = 0, \quad \widehat{A}_m = 0. \tag{69}$$

The conformal factor λ can be calculated by integrating the equations (53), which now read

$$\lambda^{-1}\partial_\pm\lambda \cdot \rho\partial_\pm\rho = \frac{1}{2}\rho^{-1}\partial_\pm^2\rho + \left(\partial_\pm \ln\frac{-t_2}{t_1}\right)^2. \tag{70}$$

Without any detailed calculation, we can immediately find the expression of λ if we notice that this equation is completely the same as the one that appears in the construction of a Ferrari-Ibañez metric in Ref. [11], except the extra factor 4 in the second term of the left hand side in (70). The result is

$$\lambda^2 = \text{const.} \times uv \left(\frac{(1-t_1t_2)^2}{(1-t_1^2)(1-t_2^2)}\right)^4. \tag{71}$$

We have seen in this example that the problem to solve the string equations of motion can be deduced to almost the same problem arising in general relativity, if we assume some special form of the string backgrounds. Of course one could obtain more non-trivial solutions if one starts from more complicated monodromy matrices. We will not do this, but will pursue the similarity between the general relativity and string theory in the subsequent sections.

IV. EMBEDDING GENERAL RELATIVITY INTO STRING THEORY

A. Embedding Einstein Theory into O(8,8) Theory

In this section we will identify more general string backgrounds such that they satisfy the (electro-magnetic) Ernst equation. We first switch off the $\widehat{A}_\mu$ fields, i.e. work with the

O(8,8)/ O(8)×O(8) theory. We may hence consider only the upper-left 16×16 matrices. The V matrix is then reduced to

$$V = \begin{bmatrix} \widehat{E}^{-1} & 0 \\ -\widehat{B}\widehat{E}^{-1} & \widehat{E}^T \end{bmatrix}. \tag{72}$$

Making use of the relations (45)(46), we obtain

$$\begin{aligned} & P_\mu \\ &= \frac{1}{2}\begin{bmatrix} 0 & \widehat{E}^{-1}\partial\mu\widehat{E}+\partial\mu\widehat{E}\cdot\widehat{E}^{-1}+\widehat{E}^{-1}\partial_\mu\widehat{B}\cdot\widehat{E}^{-1} \\ \widehat{E}^{-1}\partial\mu\widehat{E}+\partial\mu\widehat{E}\cdot\widehat{E}^{-1}-\widehat{E}^{-1}\partial_\mu\widehat{B}\cdot\widehat{E}^{-1} & 0 \end{bmatrix}, \end{aligned} \tag{73}$$

$$\begin{aligned} & Q_\mu \\ &= \frac{1}{2}\begin{bmatrix} \widehat{E}^{-1}\partial\mu\widehat{E}-\partial\mu\widehat{E}\cdot\widehat{E}^{-1}-\widehat{E}^{-1}\partial_\mu\widehat{B}\cdot\widehat{E}^{-1} & 0 \\ 0 & \widehat{E}^{-1}\partial\mu\widehat{E}-\partial\mu\widehat{E}\cdot\widehat{E}^{-1}+\widehat{E}^{-1}\partial_\mu\widehat{B}\cdot\widehat{E}^{-1} \end{bmatrix}, \end{aligned} \tag{74}$$

where we have taken $\widehat{E}$ to be symmetric by using the local Lorentz rotation in the compactified sector. The second term in $[\cdots]$ of (50) then reads

$$\begin{aligned} & \eta^{\mu\nu}\mathrm{Tr}P_\mu P_\nu \\ &= \eta^{\mu\nu}\frac{1}{2}\mathrm{Tr}\left\{(\widehat{E}^{-1}\partial_\mu\widehat{E}+\partial_\mu\widehat{E}\cdot\widehat{E}^{-1})(\widehat{E}^{-1}\partial_\nu\widehat{E}+\partial_\nu\widehat{E}\cdot\widehat{E}^{-1})\right. \\ &\qquad \left. -\widehat{E}^{-1}\partial_\mu\widehat{B}\widehat{E}^{-1}\widehat{E}^{-1}\partial_\nu\widehat{B}\widehat{E}^{-1}\right\}, \end{aligned} \tag{75}$$

where the trace in the right hand side is the one for 8×8 matrices. Let us now assume that $\widehat{E}$ and $\widehat{B}$ are in the forms

$$\widehat{E} = \Delta^{\frac{1}{2}}X, \quad \widehat{B} = BY, \tag{76}$$

for some scalar fields Δ and B, where X is an arbitrary constant 8×8 matrix satisfying $X^2 = I_8$, and Y is an arbitrary constant 8×8 antisymmetric matrix. We normalize Y as

$$\mathrm{Tr}Y^2 = -8 \tag{77}$$

for convenience. For example let us take

$$X = I_8, \quad Y = \begin{bmatrix} & I_4 \\ -I_4 & \end{bmatrix}. \tag{78}$$

Substituting (76) and (78) into (73), we then find

$$P_\mu = \frac{1}{2} \begin{bmatrix} \begin{matrix} & \\ & 0 & \\ & \end{matrix} & \begin{matrix} \Delta^{-1}\partial_\mu \Delta I_4 & \Delta^{-1}\partial_\mu B I_4 \\ -\Delta^{-1}\partial_\mu B I_4 & \Delta^{-1}\partial_\mu \Delta I_4 \end{matrix} \\ \begin{matrix} \Delta^{-1}\partial_\mu \Delta I_4 & \Delta^{-1}\partial_\mu B I_4 \\ -\Delta^{-1}\partial_\mu B I_4 & \Delta^{-1}\partial_\mu \Delta I_4 \end{matrix} & \begin{matrix} & \\ & 0 & \\ & \end{matrix} \end{bmatrix}. \tag{79}$$

(75) reads

$$\eta^{\mu\nu} \mathrm{Tr} P_\mu P_\nu = \frac{1}{2} \cdot 8 \cdot \eta^{\mu\nu} \Delta^{-2} (\partial_\mu \Delta \partial_\nu \Delta + \partial_\mu B \partial_\nu B). \tag{80}$$

The total action (50) now has the form

$$S = \int d^2 x e^{-\Phi} \left[\sqrt{-g} R_g - 2\eta^{\mu\nu} \Delta^{-2} (\partial_\mu \Delta \partial_\nu \Delta + \partial_\mu B \partial_\nu B). \right]. \tag{81}$$

This action is "almost" identical to the SL(2,$\mathbf{R}$)/U(1) sigma model that arises in the reduction of the Einstein action from four dimensions to two. The only difference is the coupling constant of the sigma model kinetic terms, which cannot be absorbed by rescaling of fields.

This fact implies the following significant consequences. First, we observe that the both actions lead to the same equation of motion of Δ and B fields — the Ernst equation. This means that we may construct solutions of the equation of motion of $\widehat{G}$ and $\widehat{B}$ fields in a low-energy string theory from the known solutions of the Ernst equation in general relativity. Second, the conformal factor λ is determined by integrating the "Virasoro conditions" (53) (the equations of motion of the missing component of $G_{\mu\nu}$), which now reads

$$\lambda^{-1} \partial_\pm \lambda \cdot \rho \partial_\pm \rho = \frac{1}{2} \rho^{-1} \partial_\pm^2 \rho + 1 \cdot \frac{(\partial_\pm \Delta)^2 + (\partial_\pm B)^2}{\Delta^2}. \tag{82}$$

It is instructive to compare these expression with the corresponding equations of motion in general relativity, which are given by (See [11])

$$\text{(General relativity)} \qquad \lambda_{\rm gr}^{-1}\partial_\pm\lambda_{\rm gr}\cdot\rho\partial_\pm\rho = \frac{1}{2}\rho^{-1}\partial_\pm^2\rho + \frac{1}{4}\cdot\frac{(\partial_\pm\Delta)^2+(\partial_\pm B)^2}{\Delta^2}. \tag{83}$$

The difference by a factor 4 comes from the coefficient 1/2 of the second term of (50), which is 1 in the case of general relativity, times 8: the dimensions of the compactified space. Therefore, the conformal factor of the string is just the one of the general relativity to the 4, except the common explicitly ρ-dependent factor. To be more precise, they are related by

$$\frac{\lambda^2}{uv} = \left(\frac{\lambda_{\rm gr}^2}{uv}\right)^4. \tag{84}$$

in terms of the (u,v) coordinates (66).

B. Embedding Einstein-Maxwell Theory into Heterotic String Theory

In the last subsection we saw that the nonlinear sigma model associated with the reduced pure Einstein theory can be embedded into the dimensionally reduced O(8,8) theory, where all the gauge fields $\widehat{A}_\mu$ are set to be zero, as the Ernst potential contains only two degrees of freedom. We next switch on the $\widehat{A}_\mu$ fields, and will show that the nonlinear sigma model associated with the Einstein-Maxwell theory can be embedded into that of the heterotic string with such field configurations. We now consider the following forms of the fields

$$\widehat{E} = \Delta^{\frac{1}{2}}I_8,\ \ \widehat{B} = BY,\ \ \widehat{A} = [Z \ \ W]\,. \tag{85}$$

Here Z and W are 8×8 matrices that depends on some two scalar fields. After some calculation we find

$$P_\mu$$

$$
= \begin{bmatrix}
0 & \begin{array}{c} \frac{1}{2}\Delta^{-1}\partial_\mu\Delta + \frac{1}{2}\Delta^{-1}\partial_\mu BY \\ -\frac{1}{4}\Delta^{-1}\big(\partial_\mu Z\cdot Z^T - Z\partial_\mu Z^T \\ +\partial_\mu W\cdot W^T - W\partial_\mu W^T\big) \end{array} & \frac{1}{\sqrt{2}}\Delta^{-\frac{1}{2}}\partial_\mu Z & \frac{1}{\sqrt{2}}\Delta^{-\frac{1}{2}}\partial_\mu W \\
\begin{array}{c} \frac{1}{2}\Delta^{-1}\partial_\mu\Delta - \frac{1}{2}\Delta^{-1}\partial_\mu BY \\ +\frac{1}{4}\Delta^{-1}\big(\partial_\mu Z\cdot Z^T - Z\partial_\mu Z^T \\ +\partial_\mu W\cdot W^T - W\partial_\mu W^T\big) \end{array} & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}}\Delta^{-\frac{1}{2}}\partial_\mu Z^T & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}}\Delta^{-\frac{1}{2}}\partial_\mu Z^T & 0 & 0 & 0
\end{bmatrix},
\tag{86}
$$

$$
Q_\mu
$$

$$
= \begin{bmatrix}
\begin{array}{c} -\frac{1}{2}\Delta^{-1}\partial_\mu BY \\ +\frac{1}{4}\Delta^{-1}\big(\partial_\mu Z\cdot Z^T - Z\partial_\mu Z^T \\ +\partial_\mu W\cdot W^T - W\partial_\mu W^T\big) \end{array} & 0 & 0 & 0 \\
0 & \begin{array}{c} \frac{1}{2}\Delta^{-1}\partial_\mu BY \\ -\frac{1}{4}\Delta^{-1}\big(\partial_\mu Z\cdot Z^T - Z\partial_\mu Z^T \\ +\partial_\mu W\cdot W^T - W\partial_\mu W^T\big) \end{array} & \frac{1}{\sqrt{2}}\Delta^{-\frac{1}{2}}\partial_\mu Z & \frac{1}{\sqrt{2}}\Delta^{-\frac{1}{2}}\partial_\mu W \\
0 & -\frac{1}{\sqrt{2}}\Delta^{-\frac{1}{2}}\partial_\mu Z^T & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}}\Delta^{-\frac{1}{2}}\partial_\mu W^T & 0 & 0
\end{bmatrix},
\tag{87}
$$

where we omitted I_8. Let us now assume

$$
Z = W = HI_8 + EY \tag{88}
$$

for some scalar fields E and H. This special parameterization yields

$$
\begin{aligned}
\eta^{\mu\nu}\mathrm{Tr}P_\mu P_\nu = 4\eta^{\mu\nu}\Big[&\Delta^{-2}\partial_\mu\Delta\partial_\nu\Delta \\
&+\Delta^{-2}\left(\partial_\mu B + 2(E\partial_\mu H - H\partial_\mu E)\right)\left(\partial_\nu B + 2(E\partial_\nu H - H\partial_\nu E)\right) \\
&+4\Delta^{-1}(\partial_\mu E\partial_\nu E + \partial_\mu H\partial_\nu H)\Big].
\end{aligned}
\tag{89}
$$

This is the SU(2,1)/S(U(2)×U(1)) nonlinear sigma model, which is known to arise as a result of the dimensional reduction of the Einstein-Maxwell theory [26,27]. The particular configuration (85) with (88) thus satisfies the electro-magnetic Ernst equation. The choice (88) is not the unique. We would like to emphasize here that one may construct a low-energy

string solution from every known solution of the electro-magnetic Ernst equation in general relativity [28,29].

C. Modular Transformation as T Duality

It is naturally expected that the modular transformation discussed in Sect.2 can be regarded as a special T duality. We will show that this is the case [4]. The T-duality group has been shown to acts on V as [10]

$$V \mapsto UV, \tag{90}$$

where

$$U \in \mathrm{O}(8,24;\mathbf{Z})_L \equiv \left\{ U \,\middle|\, U^T L U = L \right\}. \tag{91}$$

A matrix U in the form

$$U = \begin{bmatrix} U_1 & U_2 & \\ U_3 & U_4 & \\ & & I_{16} \end{bmatrix} \tag{92}$$

belongs to $\mathrm{O}(8,24;\mathbf{Z})_L$ if

$$\begin{aligned} &U_3^T U_1 + U_1^T U_3 = 0, U_3^T U_2 + U_1^T U_4 = I_8, \\ &U_4^T U_1 + U_2^T U_3 = I_8, U_4^T U_2 + U_2^T U_4 = 0. \end{aligned} \tag{93}$$

Let us now consider the subgroup

---

[4] Although the Ernst equation is originally an equation of the Ernst potential, which is defined in the Ehlers picture (11), we may move on to the Matzner-Misner picture (14) through the Kramer-Neugebauer involution (17). Because of this, we may well identify either modular transformation as T duality.

$$\left\{ U = \begin{bmatrix} aI_8 & bY & \\ cY^T & dI_8 & \\ & & I_{16} \end{bmatrix} \middle| \; a,b,c,d \in \mathbf{Z},\;\; ad-bc=1 \right\}, \tag{94}$$

where Y is given by (78). It is an elementary exercise to check that this is indeed a subgroup of $\mathrm{O}(8,24;\mathbf{Z})_L$, and that this is isomorphic to SL(2,**Z**).

We again set $\widehat{A}_\mu = 0$. It is now almost clear that this SL(2,**Z**) corresponds to the integral Ehlers group of the embedded reduced Einstein gravity. Indeed, the V matrix is in the form

$$V = \begin{bmatrix} \Delta^{-\frac{1}{2}} I_8 & 0 & \\ \Delta^{-\frac{1}{2}} BY^T & \Delta^{\frac{1}{2}} I_8 & \\ & & I_{16} \end{bmatrix}. \tag{95}$$

Acting U of the form (94) on V from the left makes V deviate from the triangular gauge, so that a compensating gauge transformation from the right is necessary. This is the well-known nonlinear realization [30]. To be concrete, let

$$U = \begin{bmatrix} dI_8 & cY & \\ bY^T & aI_8 & \\ & & I_{16} \end{bmatrix}, \quad a,b,c,d \in \mathbf{Z},\;\; ad-bc=1, \tag{96}$$

and

$$h = \begin{bmatrix} pI_8 & -qY & \\ qY^T & pI_8 & \\ & & I_{16} \end{bmatrix}, \tag{97}$$

$$p = \frac{cB+d}{[(cB+d)^2+(c\Delta)^2]^{\frac{1}{2}}}, \quad q = \frac{c\Delta}{[(cB+d)^2+(c\Delta)^2]^{\frac{1}{2}}}, \tag{98}$$

then

$$V \mapsto UVh \tag{99}$$

causes the integral Ehlers SL(2,**Z**) (26). It is easy to check that $U_0^T h U_0$ is an element of O(8)×O(24). This establishes the embedding of the Ehlers SL(2,**Z**) into the T duality.

Obviously, if one replaces the V matrix by its image mapped by the Kramer-Neugebauer involution, one would get an embedding of the Matzner-Misner SL(2,**Z**). Thus either of the modular groups discussed in the previous subsection may be identified as a subgroup of the T duality.

D. Integral Matzner-Misner and Ehlers Symmetry as T- and S-duality

We will now embed the Matzner-Misner SL(2,**Z**) into the T-duality in this subsection, and will show that the Ehlers SL(2,**Z**) then acts as the S-duality on the four-dimensional sector.

We start again from the ten-dimensional action (28). Following Ref. [22] we introduce the fields $\widehat{G}'_{m'n'}$, $\widehat{B}'_{m'n'}$, Φ', $G'_{\mu'\nu'}$ and $B'_{\mu'\nu'}$ defined by

$$\begin{aligned}
&\widehat{G}'_{m'n'} = G^{(10)}_{m'+3,n'+3},\ \widehat{B}'_{m'n'} = B^{(10)}_{m'+3,n'+3},\\
&A'^{(m')}_{\mu'} = \frac{1}{2}\widehat{G}'^{m'n'}G^{(10)}_{n'+3,\mu'},\\
&A'^{(m'+6)}_{\mu'} = \frac{1}{2}B^{(10)}_{m'+3,\mu'} - \widehat{B}'_{m'n'}A'^{(n')}_{\mu'}, && (100)\\
&G'_{\mu'\nu'} = G^{(10)}_{\mu'\nu'} - G^{(10)}_{m'+3,\mu'}G^{(10)}_{n'+3,\nu'}\widehat{G}'^{m'n'},\\
&B'_{\mu'\nu'} = B^{(10)}_{\mu'\nu'} - 4\widehat{B}'_{m'n'}A'^{(m')}_{\mu'}A'^{(n')}_{\nu'} - 2(A'^{(m')}_{\mu'}A'^{(m'+6)}_{\nu'} - A'^{(m')}_{\nu'}A'^{(m'+6)}_{\mu'}),\\
&\Phi' = \Phi^{(10)} - \frac{1}{2}\ln\det\widehat{G}', && (101)
\end{aligned}$$

where we take z^M with $M = 0, \ldots, 3$ as the four-dimensional space-time coordinates $x'^{\mu'}$ ($\mu' = 0, \ldots, 3$), and z^M with $M = 4, \ldots, 9$ as the coordinates $y'^{m'}$ with $m' = 1, \ldots, 6$ of the six-torus [5]. The $U(1)^{16}$ gauge fields $A^{(10)}_M$ are already set to zero. In terms of these fields the action (28) is reduced, by dropping the $z^{m'+3}$ dependence, to [22]

$$S = \int d^4x'\sqrt{-G'}e^{-\Phi'}\left[R_{G'} + G'^{\mu'\nu'}\partial_{\mu'}\Phi'\partial_{\nu'}\Phi'\right.$$

---

[5] In this paper we use the primed indices and quantities for the decomposition into four and six dimensions. They appear only in this subsection.

$$
\begin{aligned}
&-\frac{1}{12}G'^{\mu'_1\mu'_2}G'^{\nu'_1\nu'_2}G'^{\rho'_1\rho'_2}H'_{\mu'_1\nu'_1\rho'_1}H'_{\mu'_2\nu'_2\rho'_2}\\
&-G'^{\mu'_1\mu'_2}G'^{\nu'_1\nu'_2}F'^{(a')}_{\mu'_1\nu'_1}(L'M'L')_{a'b'}F'^{(b')}_{\mu'_2\nu'_2}\\
&+\frac{1}{8}G^{\mu'\nu'}\mathrm{Tr}(\partial_{\mu'}M'L'\partial_{\nu'}M'L')\Big],
\end{aligned} \tag{102}
$$

$$
F'^{(a')}_{\mu'\nu'} = \partial_{\mu'}A'^{(a')}_{\nu'} - \partial_{\nu'}A'^{(a')}_{\mu'} \quad (a' = 1, \ldots, 12),
$$

$$
H'_{\mu'\nu'\rho'} = \left(\partial_{\mu'}B'_{\nu'\rho'} + 2A'^{(a')}_{\mu'}L'_{a'b'}F'^{(b')}_{\nu'\rho'}\right) + \text{cyclic permutations}, \tag{103}
$$

where

$$
L' = \begin{bmatrix} & I_6 \\ I_6 & \end{bmatrix}, \quad M' = \begin{bmatrix} \widehat{G}'^{-1} & \widehat{G}'^{-1}\widehat{B}' \\ -\widehat{B}'\widehat{G}'^{-1} & \widehat{G}' - \widehat{B}'\widehat{G}'^{-1}\widehat{B}' \end{bmatrix}. \tag{104}
$$

To embed the Matzner-Misner SL(2,$\mathbf{R}$) we now take the ten-dimensional string-background configurations as

$$
G^{(10)}_{MN} = \begin{bmatrix} -\lambda'^2 & & \\ & \lambda'^2 & \\ & & \frac{\rho}{\Delta}I_8 \end{bmatrix}, B^{(10)}_{MN} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & & B_x I_4 \\ & & -B_x I_4 & \end{bmatrix},
$$

$$
e^{-\Phi^{(10)}} = \left(\frac{\rho}{\Delta}\right)^{-4}\rho, \tag{105}
$$

which are the image of (31) with (49), (51), (76) by the Kramer-Neugebauer involution (17), except the conformal factor λ'. For our purpose we cyclically permute the $M = 3, \ldots, 6$ coordinates $(3,4,5,6) \mapsto (6,3,4,5)$, so that we have

$$
\begin{aligned}
&G'_{\mu'\nu'} = \begin{bmatrix} -\lambda'^2 & & & \\ & \lambda'^2 & & \\ & & \frac{\rho}{\Delta} & \\ & & & \frac{\rho}{\Delta} \end{bmatrix}, B'_{\mu'\nu'} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & & B_x \\ & & -B_x & \end{bmatrix},\\
&\widehat{G}'_{m'n'} = \begin{bmatrix} \frac{\rho}{\Delta}I_6 \end{bmatrix}, \widehat{B}'_{m'n'} = \begin{bmatrix} & B_x I_3 \\ -B_x I_3 & \end{bmatrix},\\
&H'_{\mu'\nu'\sigma'} = \partial_{\mu'}B'_{\nu'\sigma'} + \partial_{\nu'}B'_{\sigma'\mu} + \partial_{\sigma'}B'_{\mu'\nu'},\\
&F'^{(a')}_{\mu'\nu'} = 0 \quad (a' = 1, \ldots, 12),
\end{aligned} \tag{106}
$$

and

$$e^{-\Phi'} = \Delta. \tag{107}$$

The S-duality is expressed as the invariance of the equations of motion derived from (102) under the integral fractional linear transformations of the complex potential

$$\Lambda \equiv \Psi + ie^{-\Phi'}, \tag{108}$$

where the scalar field Ψ is defined by the relation

$$H'^{\mu'\nu'\rho'} = -(\sqrt{-g'})^{-1} e^{2\Phi'} \epsilon^{\mu'\nu'\rho'\sigma'} \partial_{\sigma'} \Psi, \tag{109}$$

$$g'_{\mu'\nu'} = e^{-\Psi'} G'_{\mu'\nu'}. \tag{110}$$

Substitution of (106) into (109) yields

$$\rho^{-1} H'_{\mu 23} = -\Delta^{-2} \epsilon^{(2)}{}_{\mu}{}^{\nu} \partial_\nu \Psi. \tag{111}$$

Comparing (111) with (12), we immediately find that

$$\Psi = B \tag{112}$$

up to a constant. Hence Λ defined in (108) is nothing but the Ernst potential. Thus we see that the Ehlers SL(2,**Z**) acts as the S-duality on the four-dimensional sector if the Matzner-Misner SL(2,**Z**) is embedded into the T-duality.

V. COLLIDING STRING WAVE SOLUTIONS

A. Ferrari-Ibañez Type Series of Solutions

We are now in a position to utilize our results to derive some explicit solutions of classical string theory. Let us first consider an infinite series of $\widehat{B}_{\mu\nu}, \widehat{A}_\mu = 0$ solutions recast from the Ferrari-Ibañez metrics [31].

The Ferrari-Ibañez metrics are given by

$$
\begin{aligned}
ds^2_{\mathrm{FI}}(n) = -\rho^{\frac{n^2-1}{2}}(1-\xi)^{1+n}(1+\xi)^{1-n}\left(\frac{d\xi^2}{1-\xi^2}-\frac{d\eta^2}{1-\eta^2}\right) \\
+\rho^{1-n}\frac{1+\xi}{1-\xi}dx^2+\rho^{1+n}\frac{1-\xi}{1+\xi}dy^2,
\end{aligned}
\tag{113}
$$

where

$$
\begin{aligned}
\xi &\equiv u\sqrt{1-v^2}+v\sqrt{1-u^2}, \\
\eta &\equiv u\sqrt{1-v^2}-v\sqrt{1-u^2}, \\
\rho^2 &= (1-\xi^2)(1-\eta^2).
\end{aligned}
\tag{114}
$$

The metric $ds^2_{\mathrm{FI}}(n+2)$ can be obtained from $ds^2_{\mathrm{FI}}(n)$ by successive applications of the Ehlers transformation (27) and a change of the coordinates $x \mapsto y$, $y \mapsto -x$. Because of the fact that the latter interchanges the a- and b-cycles on the "compactified" two-torus, and the discussion in Subsect.II B, we see that the infinite series of metrics are generated from the first two by alternately applying the modular S-transformations in the Ehlers and the Matzner-Misner pictures. The n dependence of the conformal factor reflects the existence of the central charge [9] of the Kac-Moody algebra. The embedded series of string backgrounds thus cannot be reached each other only by T-duality transformations.

The Ernst potential of this metric is

$$
\mathcal{E}_{\mathrm{FI}}(n) = \Delta_{\mathrm{FI}}(n) = \rho^{1+n}\frac{1-\xi}{1+\xi}, \quad B_{\mathrm{FI}}(n) = 0. \tag{115}
$$

The conformal factor of reads

$$
\lambda_{\mathrm{FI}}(n)^2 = \rho^{\frac{(n+1)^2}{2}}(1-\xi)^{n+2}(1+\xi)^{-n}\frac{8uv}{\xi^2-\eta^2}. \tag{116}
$$

Hence the conformal factor of the string solutions can be derived by using (84):

$$
\lambda^2 = uv\left(\frac{\lambda_{\mathrm{FI}}(n)^2}{uv}\right)^4. \tag{117}
$$

Substituting these data into (31), (49), (51) and (76), we get an infinite series of the classical string solutions corresponding to the Ferrari-Ibañez metrics.

B. $\widehat{B}_{\mu\nu} \neq 0$ Solution from a Non-collinearly Polarized Wave

We will next construct a less trivial, $\widehat{B}_{\mu\nu} \neq 0$ solution from a non-collinearly polarized wave, the Nutku-Halil metric [32,33]

$$\begin{aligned} ds^2_{\rm NH} &= -N_0 \rho^{-\frac{1}{2}} \left(\frac{d\xi^2}{1-\xi^2} - \frac{d\eta^2}{1-\eta^2} \right) \\ &\quad + \frac{\rho}{N_0} \left(|1 + p\xi + iq\eta|^2 dx^2 + 4q\eta dx dy + |1 - p\xi - iq\eta|^2 dy^2 \right) \\ &= -N_0 \rho^{-\frac{1}{2}} \left(\frac{d\xi^2}{1-\xi^2} - \frac{d\eta^2}{1-\eta^2} \right) + \frac{\rho}{N_0} \left[\frac{N_0^2}{(1-p\xi)^2 + q^2\eta^2} dx^2 \right. \\ &\quad \left. + \left((1-p\xi)^2 + q^2\eta^2 \right) \left(dy + \frac{2q\eta}{(1-p\xi)^2 + q^2\eta^2} dx \right)^2 \right], \end{aligned} \tag{118}$$

where

$$N_0 = 1 - p^2\xi^2 - q^2\eta^2, \quad p^2 + q^2 = 1. \tag{119}$$

The Ernst potential $\mathcal{E}_{\rm NH}$ is known to be [34][6]

$$\mathcal{E}_{\rm NH} = -\Xi\Pi + \frac{2(p\Pi - iq\Xi - \xi\Pi)}{p\Xi - iq\Pi}, \quad \Xi \equiv (1-\xi^2)^{\frac{1}{2}}, \ \ \Pi \equiv (1-\eta^2)^{\frac{1}{2}}. \tag{120}$$

Hence

$$\Delta_{\rm NH} = \frac{\rho}{N_0} \left((1-p\xi)^2 + q^2\eta^2 \right), \tag{121}$$

$$B_{\rm NH} = \frac{2q}{N_0} \left(p(\xi^2 - \eta^2) - \xi(1-\eta^2) \right). \tag{122}$$

The conformal factor can be found as

$$\lambda^2_{\rm NH} = \rho^{\frac{1}{2}} \left((1-p\xi)^2 + q^2\eta^2 \right) \frac{8uv}{\xi^2 - \eta^2}. \tag{123}$$

In the collinear $(q \to 0, \ p \to 1)$ limit, (118) is reduced to the $n = 0$ Ferrari-Ibañez metric. Using (84), we can similarly obtain a solution with nonzero anti-symmetric tensor field.

---

[6]Since the imaginary part of the Ernst potential may be shifted by a constant, different expressions appear in some literature.

C. $\widehat{B}_{\mu\nu}, \widehat{A}_\mu \neq 0$ Solution

The final example is a $\widehat{B}_{\mu\nu}, \widehat{A}_\mu \neq 0$ solution, which we can construct from the Einstein-Maxwell generalization of the Nutku-Halil metric [29]. In this case the Ernst and the electromagnetic Ernst potentials are simply given, in terms of the Ernst potential of the Nutku-Halil metric $\mathcal{E}_{\rm NH}$, by

$$\Delta_{\rm G} + iB_{\rm G} = \frac{\mathcal{E}_{\rm NH}}{\psi_G \overline{\psi}_{\rm G}}, \tag{124}$$

$$E_{\rm G} + iH_{\rm G} = \frac{(e+ib)\mathcal{E}_{\rm NH}}{\psi_{\rm G}}, \tag{125}$$

$$\psi_{\rm G} \equiv 1 + (e^2 + b^2)\mathcal{E}_{\rm NH}, \tag{126}$$

where e, b are arbitrary real parameters. The conformal factor is the same as that of the Nutku-Halil metric

$$\lambda_{\rm G}^2 = \lambda_{\rm NH}^2. \tag{127}$$

Using these data in (85) and (88), we obtain a string wave solution with non-vanishing $\widehat{B}_{\mu\nu}, \widehat{A}_\mu$.

VI. SUMMARY

We have shown that the non-linear sigma models of the dimensionally reduced Einstein and Einstein-Maxwell theories can be diagonally embedded into the two-dimensional effective heterotic string theory. Consequently, the embedded string backgrounds satisfy the (electro-magnetic) Ernst equation. In the pure Einstein theory, the Matzner-Misner SL(2, **R**) can be viewed as a change of conformal structure of the compactified flat two-torus, and in particular the integral ones are the modular transformations. Hence the Ehlers SL(2, **R**) and SL(2, **Z**) act similarly on another torus whose conformal structure is induced through the Kramer-Neugebauer involution. If the Matzner-Misner SL(2,**Z**) is embedded as a special T-duality, then the Ehlers acts as the S-duality on the four-dimensional sector. Using this embedding we constructed some colliding string wave solutions.

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NOTE ADDED

After completing the manuscript, the author became aware of Refs. [35], which have some overlap with the results of Sect.IV of this paper.

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